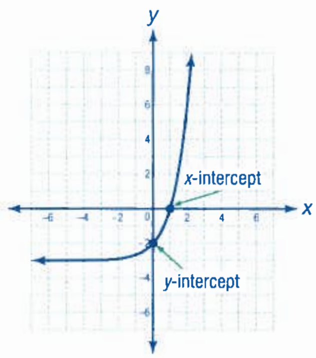
LESSON 12 Key Features of Functions

Intercepts and End Behavior

UNDERSTAND The graphs and tables of functions contain various key features. These key features are often important for understanding functions and using them to solve problems.

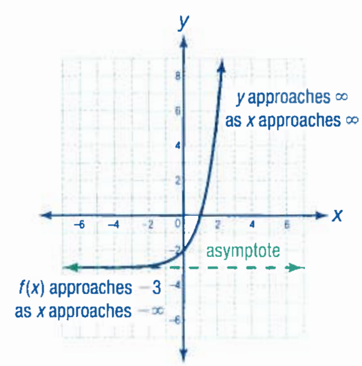
The **x-intercept** of a function is the point $(a, 0)$ at which the graph intersects the x-axis. The **y-intercept** is the point $(0, b)$ at which the graph intersects the y-axis. In the graph of $f(x) = 3^x - 3$ shown, the x-intercept is (1, 0) and the y-intercept is (0, −2).



x	$f(x)$	
−2	$-2\frac{8}{9}$	
−1	$-2\frac{2}{3}$	
0	−2	← y-intercept
1	0	← x-intercept
2	6	

You can locate the x-intercept in a table by finding the row whose y-value is 0. The y-intercept is in the row whose x-value is 0.

Functions can also be described in terms of their **end behavior**. In the graph of $f(x) = 3^x - 3$, look at the arrows on each end of the graph. The arrow on the right end of the curve shows that as x increases, y also continuously increases. Since the value of y is continuously increasing, this function has no **maximum** value. The arrow on the left end of the curve shows that as x decreases (becomes more negative), y approaches but never reaches −3. This line that the graph approaches but never touches is called the **asymptote** of the function. Since the graph asymptotically approaches the line $y = -3$ but never intersects it, the function has no **minimum** value.



If enough values are listed in a table, you can estimate end behavior based on the values of $f(x)$. Starting from the top of the second column and moving down, notice that the value of $f(x)$ gets larger and larger. Starting from the bottom of the column and moving up, notice that $f(x)$ gets smaller (more negative) but never passes −3.

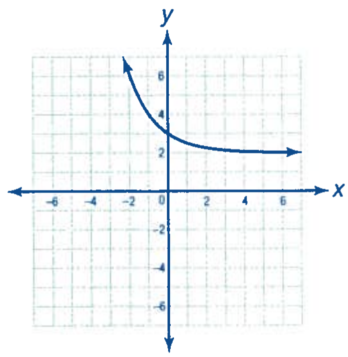
x	$f(x)$
−3	$-2\frac{26}{27}$
−2	$-2\frac{8}{9}$
−1	$-2\frac{2}{3}$
0	−2
1	0
2	6
3	24

↓ $f(x)$ increases without bound

↑ $f(x)$ decreases toward −3

Connect

The function $f(x) = \left(\frac{1}{2}\right)^x + 2$ is graphed below.



Identify the function's intercepts and describe its end behavior.

1

Find the x- and y-intercepts of the function.

Where does the graph intersect the x-axis?

The graph never intersects the x-axis, so the function does not have an x-intercept.

Where does the graph intersect the y-axis?

The graph intersects the y-axis at (0, 3). The function's y-intercept is (0, 3).

2

Describe the end behavior of the function.

What happens to the graph as x-values approach $-\infty$?

As x-values approach $-\infty$, y-values increase toward ∞.

What happens to the graph as x-values approach ∞?

As x-values approach ∞, y-values decrease toward 2.

This means that the function has an asymptote of $y = 2$.

TRY

Does the function have a minimum and a maximum? If so, what are they? If not, why not?

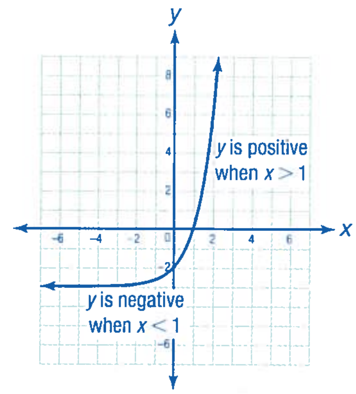
Intervals of Functions

UNDERSTAND Remember that a function's domain is the set of all possible inputs. For a function such as $f(x) = 3^x - 3$, the domain is the interval on the x-axis on which the function is defined, in which the graph exists. The range of a function is the interval on the y-axis containing all possible outputs.

Interval notation can be used to represent an interval. In interval notation, the end values of an interval are listed as a pair separated by a comma. A bracket beside a value means that it is included in the interval, while a parenthesis means that it is not. For example, the domain $[0, 5)$ is equivalent to $0 \le x < 5$.

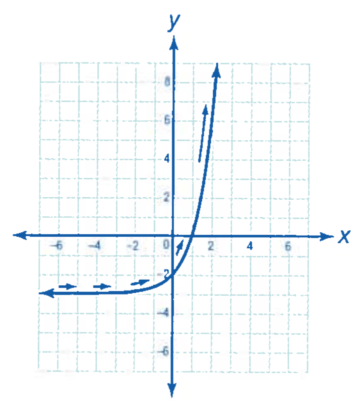
The domain can be broken up into smaller intervals that share a certain characteristic. For example, it can be useful to divide the domain into sections in which the value of $f(x)$ is positive and sections where it is negative.

Look again at a graph of the function $f(x) = 3^x - 3$. Determine the intervals where y is positive and where y is negative. The value of y is negative when $x < 1$. The value of y is positive when $x > 1$. Using interval notation, y is negative on the interval $(-\infty, 1)$ and positive on the interval $(1, \infty)$.



The domain can also be divided into sections where the value of $f(x)$ is increasing from left to right and where it is decreasing from left to right.

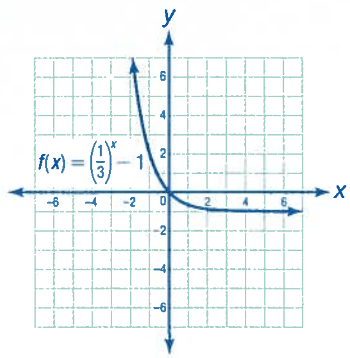
Look at the graph. From left to right, the graph is always curving upward. The value of y is always increasing as the value of x increases. This is true across the entire domain, from negative infinity $(-\infty)$ to positive infinity (∞). The function is always increasing. In other words, the interval of increase is $(-\infty, \infty)$.



Usually, these intervals can also be determined from tables by looking at the values in the $f(x)$ column.

Connect

The graph below represents an exponential function f. The table below represents a linear function g.



$g(x) = -\frac{1}{3}x$	
x	$g(x)$
−6	2
−3	1
0	0
3	−1
6	−2

Compare and contrast functions f and g by using these key features: domain; range; intervals of increase and decrease; and positive and negative intervals.

1 Identify the domain and range.

The end behavior of the graph of f shows that it extends indefinitely both left and right. Thus, its domain is all real numbers, or the interval $(-\infty, \infty)$.

Since f has an asymptote of $y = -1$, its range is $y > -1$, or the interval $(-1, \infty)$.

The table for function g does not list all values of x or $g(x)$, but it also does not give evidence of any boundaries (such as an asymptote). Since g is a linear function, without other information, you may assume that the domain and range are all real numbers.

2 Compare the intervals of increase and decrease.

The graph of f continuously curves downward. So, f is always decreasing.

The table for function g shows that as x-values increase, $g(x)$-values decrease, so g is also a decreasing function.

Both functions are decreasing across their entire domains. The interval of decrease is $(-\infty, \infty)$ for both functions.

3 Compare positive intervals and negative intervals for the functions.

The graph of f intercepts the x-axis at (0, 0). The third row of values in the table shows that g also has an x-intercept of (0, 0). The functions are always decreasing.

Functions f and g are both positive when $x < 0$, on the interval $(-\infty, 0)$, and negative when $x > 0$, on the interval $(0, \infty)$.

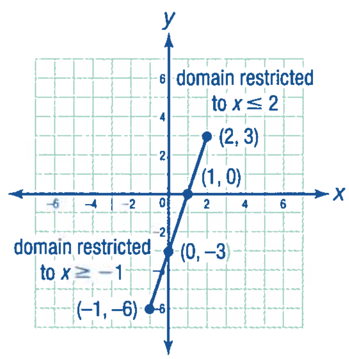
CHECK

Graph function g on the same grid as f. Compare and contrast the two graphs to check the answers on this page.

EXAMPLE A The domain of a linear function is $\{-1 \leq x \leq 2\}$. The function has an x-intercept at (1, 0) and a y-intercept at (0, −3). Graph the function. Then identify the maximum, minimum, range, and intervals of increase and decrease for the function.

1 Graph the function, paying attention to the restricted domain.

Plot the intercepts. Draw a line through the intercepts, but do not extend it to the left of −1 or to the right of 2 on the x-axis.



2 Describe the function's minimum, maximum, and range.

When the domain is restricted to $\{-1 \leq x \leq 2\}$, the lowest point on the graph is at (−1, −6). Thus, the minimum y-value is −6.

The highest point on the graph is at (2, 3). Thus, the maximum y-value is 3.

The range is all values of y greater than or equal to the minimum, −6, and less than or equal to the maximum, 3. This can be represented as $\{-6 \leq y \leq 3\}$.

3 Identify intervals of increase or decrease.

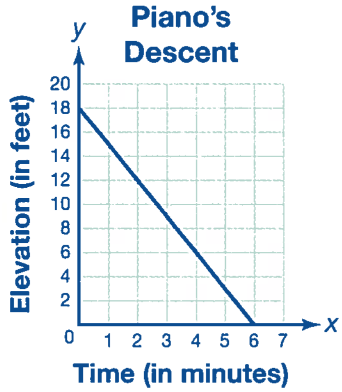
The line segment slants up from left to right, so the function is always increasing.

The value of y increases across the entire domain. The function increases on the interval $\{-1 \leq x \leq 2\}$.

TRY

Identify intervals where the linear function graphed above is positive and where it is negative.

EXAMPLE B A piano is being lowered from an apartment that is 18 feet above the sidewalk. The piano descends at a constant rate. The piano's elevation over time is represented by the linear function graphed below. Identify and interpret the key features of the graph.



1 Identify and interpret the domain.

The graph is shown to exist on the domain [0, 6]. This domain contains the minutes over which the piano is being lowered.

2 Identify and interpret intervals of increase and decrease.

The function is decreasing for the entire domain. This means that the piano's elevation is always decreasing.

The function has no interval of increase. This makes sense because the piano is always being lowered and never being raised.

3 Identify and interpret the intercepts.

The y-intercept, (0, 18), represents the piano's initial elevation of 18 feet.

The x-intercept, (6, 0), shows that it takes 6 minutes for the piano to reach the sidewalk, at an elevation of 0 feet.

TRY

What is the range for this function? What does it represent in the problem?

Practice

Rewrite each domain in interval notation.

1. $\{5 < x < 100\}$ ____________________

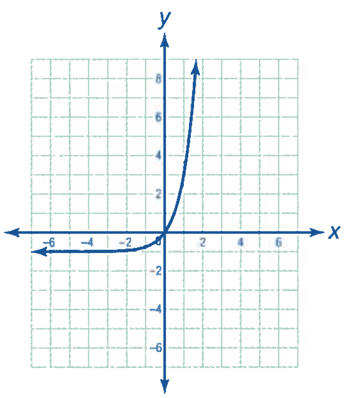
2. $\{x \geq 0\}$ ____________________

3. {all real numbers} ____________________

> REMEMBER A bracket means include the value, and a parenthesis means exclude the value.

For each graph, determine whether the function is increasing or decreasing. Identify the interval of increase or decrease.

4.



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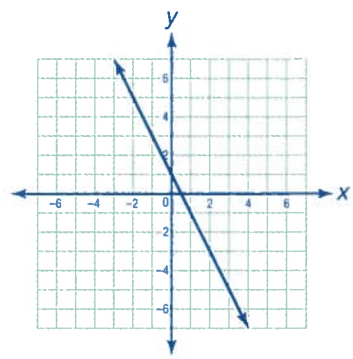
5.



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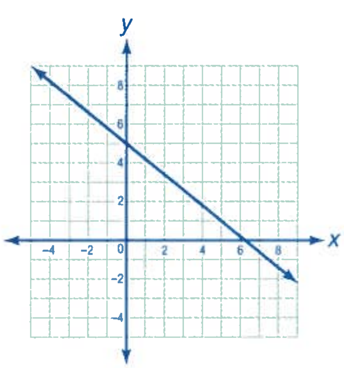
> HINT Does the graph curve (or slant) upward or downward?

Identify the intercepts of the given function.

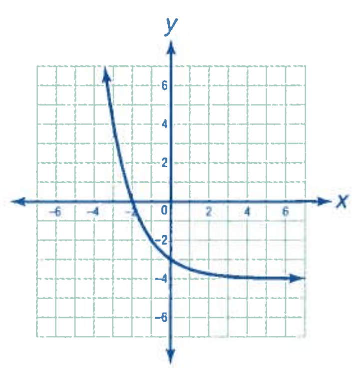
6.



x-intercept: ________________

y-intercept: ________________

7.



x-intercept: ________________

y-intercept: ________________

Identify the intercepts of the given function.

8.

x	−24	−12	0	12	24
$f(x)$	−8	−6	−4	−2	0

x-intercept: ______________

y-intercept: ______________

9.

x	−2	−1	0	1	2
$g(x)$	−9.99	−9.9	−9	0	90

x-intercept: ______________

y-intercept: ______________

Fill in each blank with an appropriate word or words.

10. A point at which a graph crosses the y-axis is a(n) ______________.

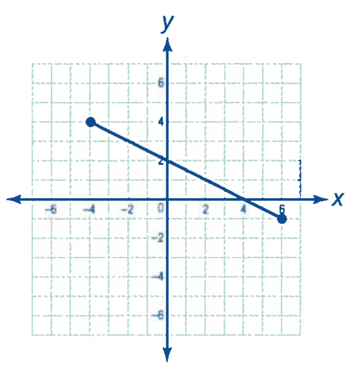
11. A function's ______________ is a line that the graph of the function approaches but never intersects.

12. The ______________ of a function describes how its $f(x)$-values change as x approaches positive infinity or negative infinity.

13. The greatest y-value on the graph of a function is the function's ______________.

Choose the best answer.

14. Which statement about this function is **not** true?



A. Its domain is $\{-4 \le x \le 6\}$.

B. Its range is $\{-1 \le y \le 4\}$.

C. It has a y-intercept at (0, 2).

D. It has a maximum of 6.

15. The table below shows some ordered pairs for an exponential function.

x	$f(x)$
−1	$-\frac{5}{6}$
0	0
1	5
2	35
3	215

Which statement about this function is **not** true?

A. Its x-intercept is the same as its y-intercept.

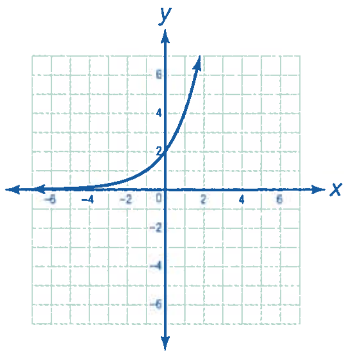
B. It is positive on the interval $(0, \infty)$.

C. It is increasing on the interval $(-\infty, \infty)$.

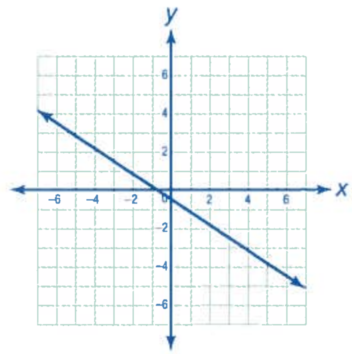
D. As x approaches $-\infty$, $f(x)$ approaches ∞.

Describe the end behavior of each function.

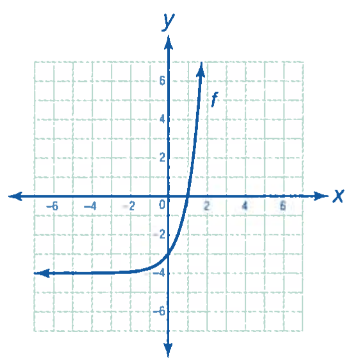
16.



17.



Use the graph and table below for questions 18–20. The graph represents exponential function *f*. The table represents some ordered pairs for linear function *g*.



x	$g(x)$
−1	−8
0	−4
1	0
2	4
3	8

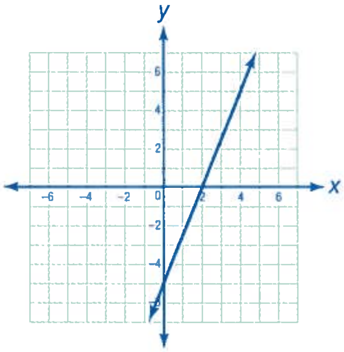
18. Compare and contrast the intercepts of the functions.

19. Compare the increasing and decreasing intervals of the functions.

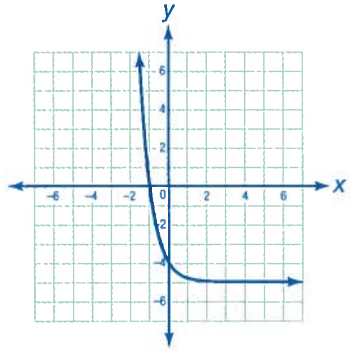
20. Compare the intervals on which the functions are positive and those on which they are negative.

For each graph, describe the intervals where the function is positive and where it is negative.

21.

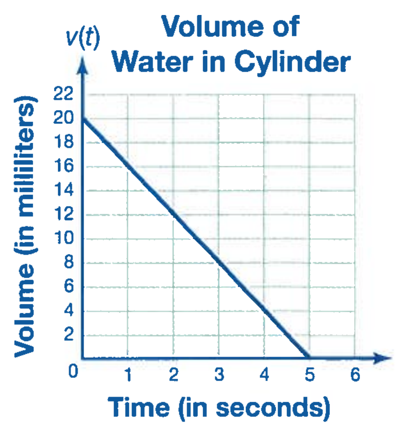


22.



Solve.

23. INTERPRET A cylinder contains 20 milliliters of water. The water begins to leak out as represented by the linear function graphed on the right. Identify the intercepts and interpret what they mean in this situation.



24. CREATE The cost of a taxi ride includes a $3 fee plus $2 for each mile traveled. So, a 1-mile ride costs $5 and a 2-mile ride costs $7. Create a graph to represent this linear function. Identify the domain for your graph and explain why you chose it.